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A REEVALUATION OF ASSUMPTIONS IN THE TURBULENT BOUNDARY-LAYER EQUATIONS WITH REGARD TO THE MOMENTUM THEOREM

A Thesis

Presented to

the Faculty of the School of Engineering

the University of Virginia

In Partial Fulfillment

of the Requirements for the Degree

Master of Aeronautical Engineering

by

Jerold Morse Bidwell

June 1951

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CHAPTER I

STATEMENT OF THE PROBLEM

I. INTRODUCTION

A general statement of the motion of a viscous fluid has been attributed to Navier and Stokes[1] but because the equations are non-linear a general solution has not been found. In investigating the Navier-Stokes equations for use in fluid flows at high Reynolds numbers, that is, when the ratio of inertia forces to viscous forces is large, Prandtl[2] noticed certain peculiarities of the flow that led to assumptions by which the equations could be simplified. He noticed that the effects of viscosity were restricted to a region immediately adjacent to the body and that elsewhere the flow could be very nearly approximated by a potential field. He called this region near the surface of the body the boundary layer, and the simplification to the Navier-Stokes equation the boundary-layer equation . In general, these boundary layers are said to be laminar or turbulent.

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[1]Prandtl, L., and Tietjens, O. G.: Fundamentals of Hydro- and Aeromechanics. McGraw-Hill Book Company, Inc., New York, 1934.

[2]Prandtl, L.: Motion of Fluids with Very Little Viscosity. NACA TM 452, 1928.

The Laminar Boundary Layer. Below a certain critical Reynolds number yet at a Reynolds number sufficiently great to maintain the concept of the boundary layer, the fluid motion is said to be laminar. In laminar boundary layers, the fluid moves in well-defined sheets or laminas. The velocity changes from the streaming velocity to zero velocity at the wall in a smooth manner. The velocity distribution through the boundary layer is maintained by viscous shear stresses that are directly proportional to the velocity gradient and by a pressure gradient that acts uniformly over the boundary layer. If the pressure gradient is known the boundary-layer equation can be solved and the boundary layer completely defined.

Solutions of the laminar boundary-layer equation have been found for only a few cases because of the extreme mathematical difficulties involved; for most cases, recourse must be taken to an approximate type of solution. If some reasonable non-dimensional shape of the velocity distribution is assumed, the change in boundary-layer momentum can be computed by utilizing the Karman momentum theorem.[3] The Karman momentum theorem is a relation between the changes in boundary-layer momentum and the external stresses, pressure

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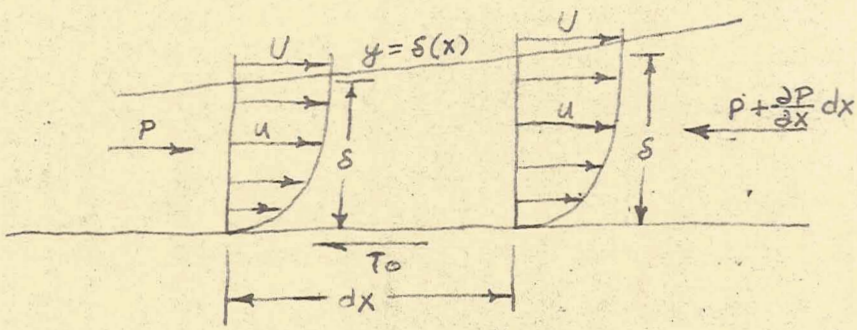
[3]Von Karman, Th.: On Laminar and Turbulent Friction. NACA TM 1092, 1946.

gradient and wall shear. This method of computation has given useful results in many cases.

The Turbulent Boundary Layer. Above some critical value of the Reynolds number, the fluid motion changes from the streaming laminar motion to a pulsating, fluctuating motion. Such a flow is generally said to be turbulent. A color filament introduced into a tube of flowing fluid remains distinct as long as the motion is laminar; when the flow becomes turbulent, the color filament breaks and fills the tube with color. The fluctuating motion carries momentum between the adjacent "layers" of fluid so that momentum interchange is not purely a viscous effect.

Because the Karman momentum theorem is, in fact, a relation between the changes in boundary layer momentum and the external stresses, pressure gradient and wall shear, it applies equally as well for turbulent boundary layers as for laminar boundary layers. Problems arise, however, in the interpretation of momentum and in the application of the theorem to experimental data.

Boundary Layer Momentum. In deriving the momentum theorem, Karman took a piece of the boundary layer



as defined by Prandtl and equated the forces and changes in momentum. The following equation was obtained:

$$\frac{\partial}{\partial t}\int_0^\delta \rho u\,dy + \frac{\partial}{\partial x}\int_0^\delta \rho u^2 dy - U\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy = -\int_0^\delta \frac{\partial P}{\partial x}dy - \tau_0$$

where $\frac{\partial}{\partial t}\int_0^\delta \rho u\,dy$ is the time rate of change of the momentum contained in the considered volume; $\frac{\partial}{\partial x}\int_0^\delta \rho u^2 dy$ is the excess of momentum leaving rear surface over the amount of momentum entering at the front; $\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy$ is the inflow per unit length of the top area, $y = \delta(x)$, so that $U\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy$ indicates the momentum entrained with this volume of fluid. The wall shearing stress τ_0 referred to unit surface is $\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$. According to the boundary layer assumptions of Prandtl, the pressure gradient is constant through the boundary layer and

$$\int_0^\delta \frac{\partial P}{\partial x}dy = \delta\frac{\partial P}{\partial x}$$

In a laminar flow the net motion is a combination of heat motion and relatively long period stream oscillations. In aerodynamics, the interest is primarily in the integrated effects of the heat motion, that is, in the contribution of heat motion to the concepts of pressure, temperature, and viscosity. If the stream motion is steady, therefore, the net motion is said to be independent of time.

In a turbulent flow the net motion is a combination of heat motion, turbulent motion, and stream motion. Reynolds showed that the periods of oscillation associated with these three phases of motion are distinguishable by means of appropriate time averages. Periods of averaging of the order of 10^{-6} second are sufficient to obtain reliable averages involving heat motion, whereas periods of averaging of the order of 10^{-1} second are sufficient to obtain reliable averages for most turbulent boundary layer motions. Motions with period longer than 10^{-1} second are associated with variations in the stream motion. The stream motion is considered steady if the average motion over 10^{-1}-second intervals are independent of the time at which such intervals are observed. In the study of turbulent motion, the fluid is considered to be a continuum and the net velocity at any point is assumed to be resolved into a fluctuating component having a period between 10^{-6} second and 10^{-1} second and a mean component which is independent of time.

If the velocities appeared in the equations of motion only to the first power, a long time average of the fluctuating component would be zero and the equations would contain terms involving only the mean velocities. Because, however, the momentum involves the squares of the velocities, momentum terms involving time averages of the squares and products of fluctuating components must be considered inasmuch as they are not zero. The expression for the Karman momentum theorem as applied to turbulent boundary layers therefore contains momentum terms for the mean flow and momentum terms involving the fluctuating components.

II. THE PROBLEM

Methods[4,5,6] for calculating the growth of a turbulent boundary layer have, in general, used an empirical equation for wall shearing stress, a known pressure distribution, an empirical equation relating the shape of the velocity distribution to the external forces, and the Karman momentum theorem

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[4]Gruschwitz, E.: Turbulent Friction Layer in Two-Dimensional Flow with Falling and Rising Pressure. Ingenieur Archiv, vol. 2, 1931, pp. 321-346.

[5]Von Doenhoff, A. E., and Tetervin, N.: Determination of General Relations for the Behavior of Turbulent Boundary Layers. NACA Report 772, 1937.

[6]Garner, H. C.: The Development of Turbulent Boundary Layers. R and M 233, British A.R.C., 1944.

to give the changes in boundary-layer momentum. The problem is determinate if it is assumed that the changes in boundary-layer momentum can be ascribed to the mean velocity alone. Calculations made with these assumptions have given good results in many instances but recently discrepancies have been noted especially in the region of separation.

Discrepancies have, also, been noted when the Karman momentum theorem was used to obtain wall shearing stress.[7,8,9,10] By assuming that the changes in boundary-layer momentum could be completely described by the mean flow, measurements were made of the mean velocity and pressure gradient and the Karman momentum theorem was used as a balance to give the wall shearing stress. This method gave values of wall shearing stress that increased in a positive

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[7]Wieghardt, K.: Uber die Wandschubspannung in turbulenten Reibungsschichten bei veranderlichem Aubendruck. U. and M. No. 6603, Z.W.B., Dec. 21, 1943.

[8]Wieghardt, K., and Tillmann, W.: Turbulent Friction Layer with Increasing Pressures. U. and M. No. 6617, Z.W.B., Nov. 20, 1944.

[9]Mangler, W.: The Behavior of the Wall Shearing Stress in Turbulent Friction Layers with Pressure Rise. U. and M. No. 3052, Z.W.B., Dec. 20, 1943.

[10]Tillmann, W.: Investigations of Some Particularities of Turbulent Boundary Layers on Plates. Rep. and Trans. No. 45, British M.A.P., Volkenrode, Mar. 15, 1946. (CGD 497 Joint Intelligence Objectives Agency, Washington, D. C.)

pressure gradient which is in contradiction to the usual concept that the wall shearing stress decreases in a positive pressure gradient and is zero at separation.

The Karman momentum theorem as given previously is strictly accurate only if the various terms represent instantaneous values. The tacit assumption has generally been made by previous authors that the Karman momentum theorem is sufficiently accurate when only mean flow quantities are inserted. In order to check this tacit assumption, it was necessary, in effect, to resolve the Karman momentum theorem into terms involving only mean flow quantities and additional terms involving only fluctuating quantities. The terms involving only mean flow quantities were put on the left-hand side of the equation and terms involving only fluctuating components were put on the right-hand side of the equation. The data given by Dryden[11] and Schubauer and Klebanoff[12] together with measurements of wall shearing stress made independently by Ludwieg and Tillmann[13] were sufficient to evaluate all the terms

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[11]Dryden, H. L.: Some Recent Contributions to the Study of Transition and Turbulent Boundary Layers. NACA TN 1168, 1947.

[12]Schubauer, G. B., and Klebanoff, P. S.: Investigation of Separation of the Turbulent Boundary Layer. NACA TN 2133, 1950.

[13]Ludwieg, H., and Tillmann, W.: Investigations of the Wall-Shearing Stress in Turbulent Boundary Layers. NACA TM 1285, 1950.

on the left-hand side. If the effect of the fluctuating terms were negligible, then the sum of all the mean flow quantities should be zero. The value of this sum obtained on the basis of available experimental information is compared with the value of the sum of the terms involving the fluctuating quantities obtained from direct measurements of these quantities.

CHAPTER II

SYMBOLS, SUBSCRIPTS AND SUPERSCRIPTS

$\frac{D}{dt}$	substantial derivative
$\underline{F}$	sum of extraneous forces acting on the fluid cube
H	boundary-layer velocity profile shape parameter, δ_*/θ
L	reference length
M	momentum loss in the boundary layer
p	absolute static pressure at any point
s	any length
t	time
u	streamwise velocity at a point in the boundary layer
U	streamwise velocity at the outer limit of the boundary layer
v	normal velocity at a point in the boundary layer
$\underline{v}$	resultant vector velocity
x	streamwise coordinate
y	normal coordinate
∇	vector operator, gradient
∇^2	scalar operator, divergence gradient
δ	boundary-layer thickness
δ_*	displacement thickness
θ	momentum thickness
μ	coefficient of viscosity
ν	coefficient of kinematic viscosity, μ/ρ

ρ fluid density

τ viscous shearing stress

Subscripts

m reference condition

o wall boundary

δ outer edge of the boundary layer

Superscripts

' turbulent fluctuation

$\overline{}$ time average

CHAPTER III

ANALYSIS

The Navier-Stokes equations for viscous compressible flows with variable viscosity can be written in vector form as

$$\frac{D\underline{v}}{dt} = \underline{F} - \frac{1}{\rho}\nabla p + \frac{1}{3}\nu\nabla(\nabla\cdot\underline{v}) + \nu\nabla^2\underline{v} - \frac{2}{3\rho}(\nabla\cdot\underline{v})\nabla\mu + \frac{2}{\rho}(\nabla\mu\cdot\nabla)\underline{v} + \frac{1}{\rho}\left[\nabla\mu\times(\nabla\times\underline{v})\right]$$

In the derivation of this equation it is assumed that the stresses are proportional to the rate of change of deformation. The pressure p at any point is defined as the arithmetic average of the normal stresses on three mutually-perpendicular planes at that point.

If now a two-dimensional flow is considered in which the viscosity and density are assumed constant and the body forces negligible, the Navier-Stokes equations become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2}$$

The associated equation of continuity for the constant density gas is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

In a turbulent flow the quantities in equations (1) and (2) are fluctuating with respect to time. After Reynolds[14]

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[14] Reynolds, O.: On the Dynamical Theory of Incompressible Viscous Fluids and the Determination of the Criterion. Philosophical Transactions of the Royal Society, vol. 2, 1895, pp. 535-577.

the motions due to heat, turbulence, and unsteady stream flow are considered to be of different order and distinguishable. A function, averaged over a time which is long compared with periods of heat motion but short compared with periods of stream motion, is associated with turbulence. A function, averaged over a time which is long compared with periods of turbulent motion, is associated with stream motion. For the present analysis, the stream motion is considered steady.

The various quantities, therefore, are considered to be composed of a mean term and a fluctuating term or

$$u = \bar{u} + u'$$

$$v = \bar{v} + v'$$

$$p = \bar{p} + p'$$

$$\rho = \bar{\rho}$$

No fluctuation density is used because the density has been assumed constant.

By substituting these quantities in equations 1, 2, and 3, and by keeping in mind that the average of a sum is the sum of the averages and that the average of a derivative is the derivative of the average, the equations for the viscous turbulent fluid are

$$\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) + \left(\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y}\right) = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} + \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2}\right) \qquad (4)$$

$$\left(\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y}\right) + \left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial y} + \nu\left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2}\right) \qquad (5)$$

and

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{6}$$

Prandtl's boundary-layer assumptions are considered to be valid for the mean flow up to separation. The boundary-layer assumptions for the mean flow are:

1. $\frac{\delta}{L}$ is much less than unity
2. L is chosen so that $\frac{x}{L}$ is of the order of magnitude unity
3. $\frac{u}{U}$ and its streamwise derivatives are of order of magnitude not greater than unity.

As a consequence of the assumptions it can be shown that the Reynolds number for which these approximations are valid is of the order $(L/\delta)^2$.

With these assumptions for the mean flow, equations (4) and (5) can be written

$$\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) + \left(\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y}\right) = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} + \nu\frac{\partial^2 \bar{u}}{\partial y^2} \tag{7}$$

$$\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial y} \tag{8}$$

From equation (8)

$$\bar{p}_o - \bar{p}_\delta = \bar{\rho}\int_0^\delta \left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy = \bar{\rho}\int_0^\delta \frac{\partial \overline{u'v'}}{\partial x}\, dy$$

because $\overline{v'v'} = 0$ at $y = \delta$

and also

$$\bar{P}_o - \bar{P} = \bar{\rho}\int_0^y \left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} \right) dy$$

then

$$\bar{P} = \bar{P}_\delta + \bar{\rho}\int_0^\delta \frac{\partial \overline{u'v'}}{\partial x}\, dy - \bar{\rho}\int_0^y \left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} \right) dy$$

and

$$\frac{1}{\bar{\rho}} \frac{\partial \bar{P}}{\partial x} = -U\frac{\partial U}{\partial x} + \frac{\partial}{\partial x}\int_0^\delta \frac{\partial \overline{u'v'}}{\partial x}\, dy - \int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} \right) dy \qquad (9)$$

because

$$\frac{1}{\bar{\rho}} \frac{\partial \bar{P}_\delta}{\partial x} = -U\frac{\partial U}{\partial x}$$

By substituting equation (9) into equation (7), the following equation can be written for a turbulent boundary layer in a two-dimensional incompressible flow:

$$\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} \right) + \left(\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} \right) = U\frac{\partial U}{\partial x}$$

$$- \frac{\partial}{\partial x}\int_0^\delta \frac{\partial \overline{u'v'}}{\partial x}\, dy + \int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} \right) dy + \nu\frac{\partial^2 \bar{u}}{\partial y^2} \qquad (10)$$

In order to describe the action of the turbulent boundary layer as a whole, equation (10) is integrated with respect to y through the boundary layer.

$$\int_0^\delta \left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) dy + \int_0^\delta \left(\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y}\right) dy =$$

$$+ \int_0^\delta U\frac{\partial U}{\partial x}\, dy - \int_0^\delta \frac{\partial}{\partial x}\int_0^\delta \frac{\partial \overline{u'v'}}{\partial x}\, dy\, dy$$

$$+ \int_0^\delta \int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy\, dy + \int_0^\delta \nu\frac{\partial^2 \bar{u}}{\partial y^2}\, dy$$

or

$$\frac{\tau_o}{\rho U^2} - \frac{\partial \theta}{\partial x} - (H+2)\frac{\theta}{U}\frac{\partial U}{\partial x} = -\frac{1}{U^2}\int_0^\delta \frac{\partial \overline{u'u'}}{\partial x}\, dy + \frac{1}{U^2}\int_0^\delta \int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy\, dy \qquad (11)$$

$$- \frac{\delta}{U^2}\int_0^\delta \frac{\partial^2 \overline{u'v'}}{\partial x^2}\, dy$$

(See Appendix)

Equation (11) is the momentum theorem for turbulent boundary layers.

The Karman momentum theorem for laminar boundary layers is given by

$$\frac{\tau_o}{\rho U^2} - \frac{\partial \theta}{\partial x} - (H+2)\frac{\theta}{U}\frac{\partial U}{\partial x} = 0$$

Whether the Karman momentum theorem can be used for turbulent boundary layers by considering only the momenta of the mean flow, can be ascertained by experimental evaluation of the momentum terms on the right side of equation (11).

CHAPTER IV

PRESENTATION OF DATA AND DISCUSSION OF RESULTS

I. MEAN-FLOW DATA

Presentation of data and discussion of accuracy. The mean-flow data of Schubauer and Klebanoff[15] have been plotted as $(U/U_m)^2$, Θ, and H as functions of x in Figures 1, 2, and 3, respectively. The original velocity data were studied in some detail and it is felt that the scatter of data is a good indication of the random errors in the measurements and in the integration procedure.

There is a systematic error in the data because pitot tube measurements give values of velocity which are in excess of the true local velocity. This effect has been reported by Goldstein[16]. The amount by which the velocity is overestimated is ε, $\varepsilon = \sqrt{\overline{u^2}+\overline{u'u'}} - \sqrt{\overline{u^2}}$. The velocity data given by Schubauer and Klebanoff have not been corrected for the turbulence but sufficient data are presented to permit this computation. The magnitude of this correction to the velocity is illustrated in Figure 4 for a point just before separation

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[15] Schubauer and Klebanoff, loc. cit.

[16] Goldstein, S.: A Note on the Measurement of Total Head and Static Pressure in a Turbulent Stream. Proc. Roy. Soc., Series A, vol. 155, no. 886, July 1, 1936, pp. 570-575.

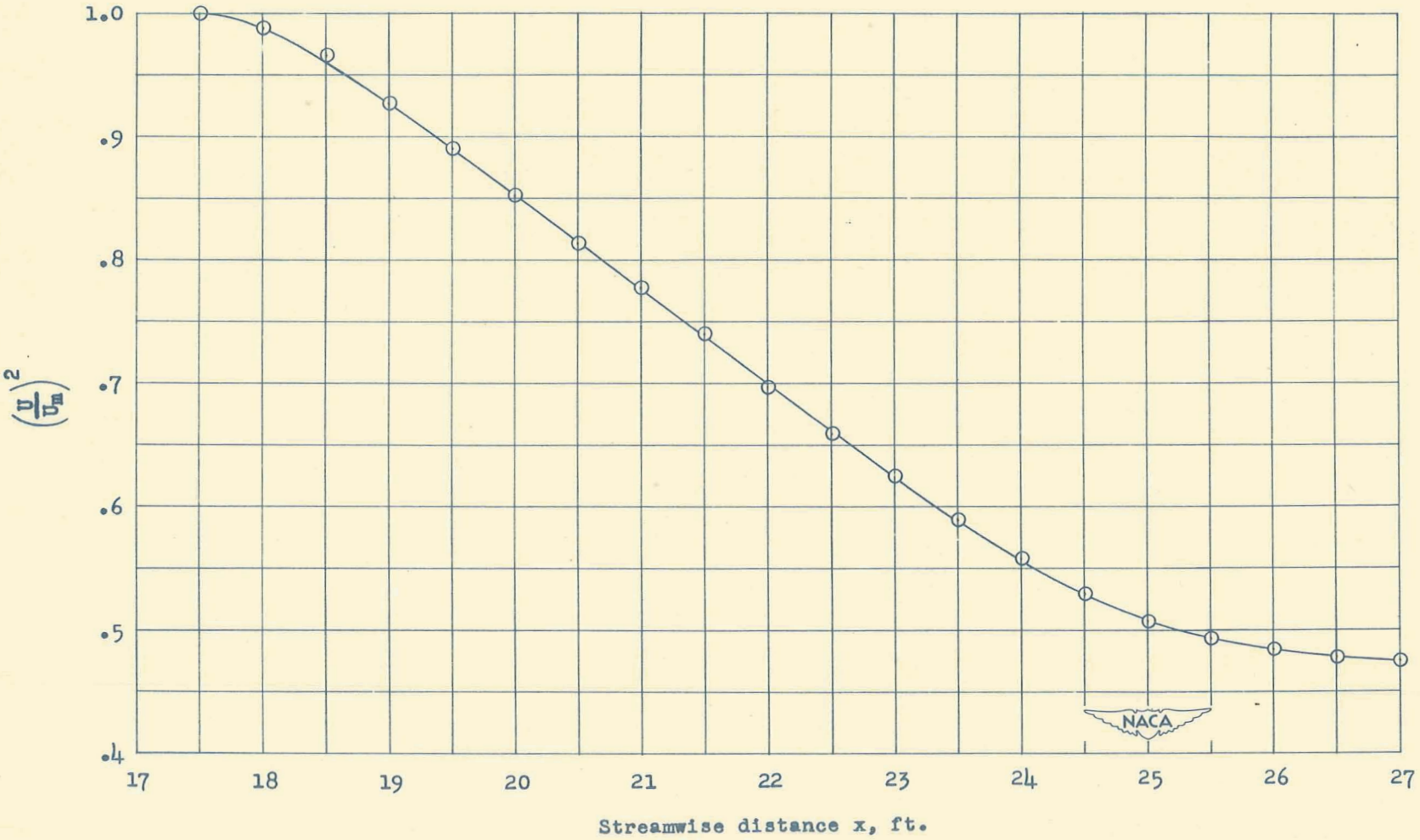

FIGURE 1

STREAMWISE VARIATION OF THE SQUARE OF THE RATIO OF THE VELOCITY OUTSIDE THE BOUNDARY LAYER TO A REFERENCE VELOCITY. DATA WERE OBTAINED FROM SCHUBAUER AND KLEBANOFF.

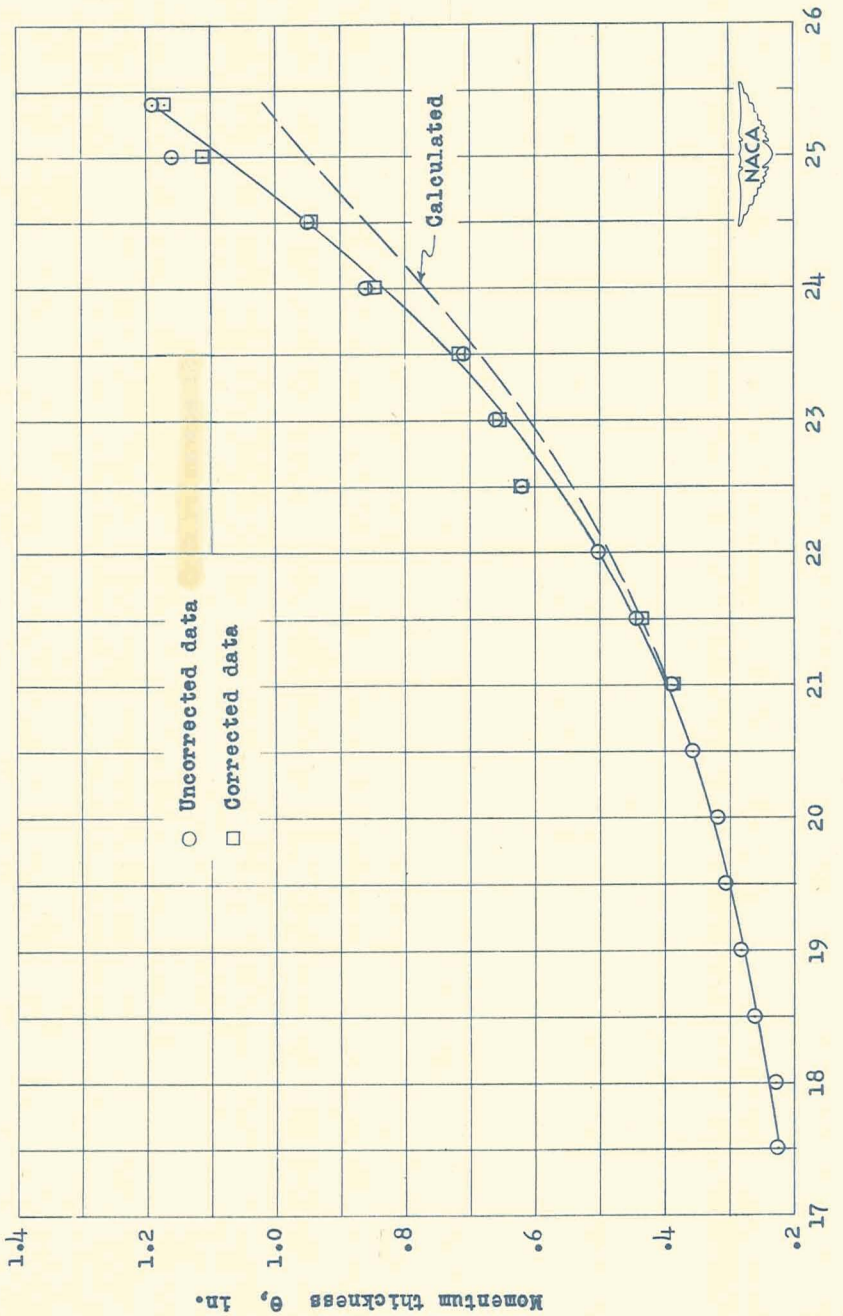

FIGURE 2

STREAMWISE VARIATION OF THE BOUNDARY-LAYER MOMENTUM THICKNESS. DATA WERE OBTAINED FROM SCHUBAUER AND KLEBANOFF.

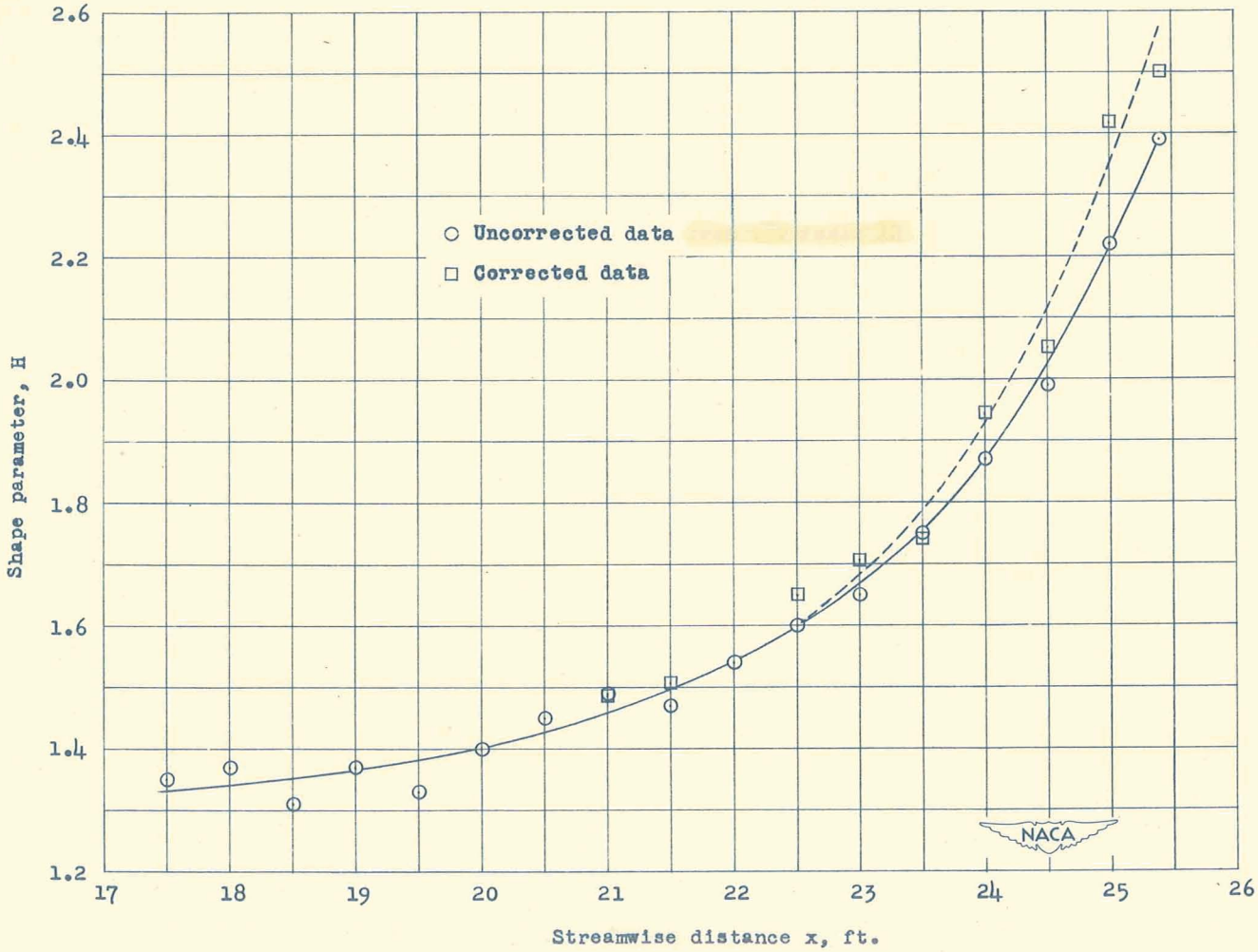

FIGURE 3

STREAMWISE VARIATION OF THE BOUNDARY-LAYER-SHAPE PARAMETER H .
DATA WERE OBTAINED FROM SCHUBAUER AND KLEBANOFF.

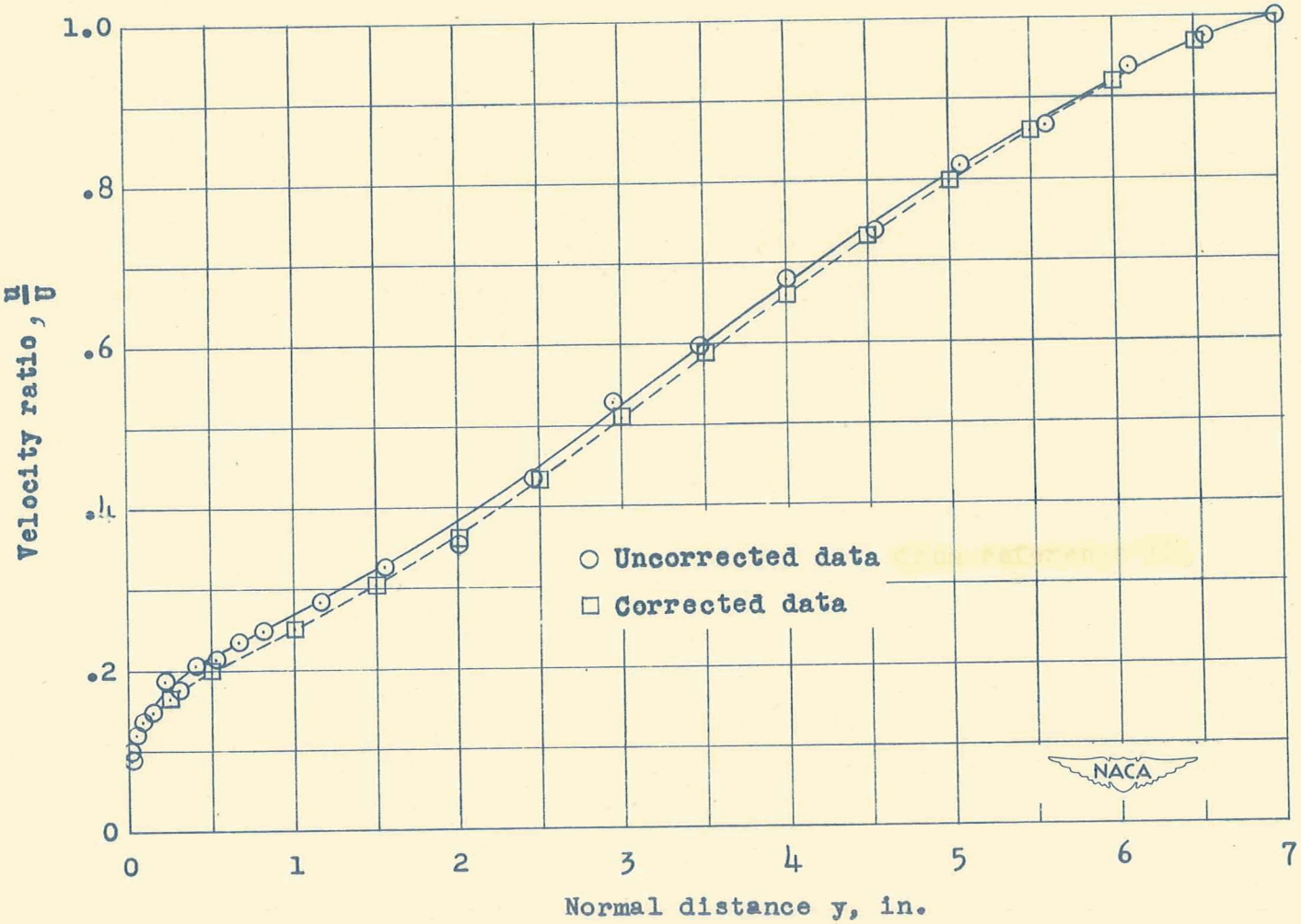

FIGURE 4

EFFECT ON BOUNDARY-LAYER VELOCITY DISTRIBUTION OF THE CORRECTION FACTOR TO THE PITOT-TUBE MEASUREMENTS. DATA WERE OBTAINED FROM SCHUBAUER AND KLEBANOFF. (x = 25.4 ft.)

($x = 25.4$ feet). The difference between the curves is somewhat greater than differences that might be indicated by random errors alone.

The velocity profiles have been corrected and integrated and the corrected values of θ are plotted in Figure 2 (page 19). At small values of x, that is less than 21.5 feet, the correction to the velocity profile was negligible. Because the correction resulted in a decrease of velocity, the displacement thickness was increased and the values of corrected H were considerably higher. The corrected values of H are plotted in Figure 3 (page 20) and re-faired with a dashed line. The corrected values of these parameters have been used in the subsequent computations.

Comparison of wall shearing stress as obtained by different methods. In Figure 5, the values of wall shearing stress estimated by Schubauer and Klebanoff are compared with the values of wall shearing stress obtained by use of the formulas of Squire and Young[17], Falkner[18], and Ludwieg and Tillmann[19]. The flat plate (zero pressure gradient) formula

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[17]Squire, H. B., and Young, A. D.: The Calculation of the Profile Drag of Aerofoils. R. and M No. 1838, British A.R.C., 1938.

[18]Falkner, V. M.: The Resistance of a Smooth Flat Plate with Turbulent Boundary Layer. Aircraft Engineering, vol. XV, No. 169, March 1943.

[19]Ludwieg and Tillman, loc. cit.

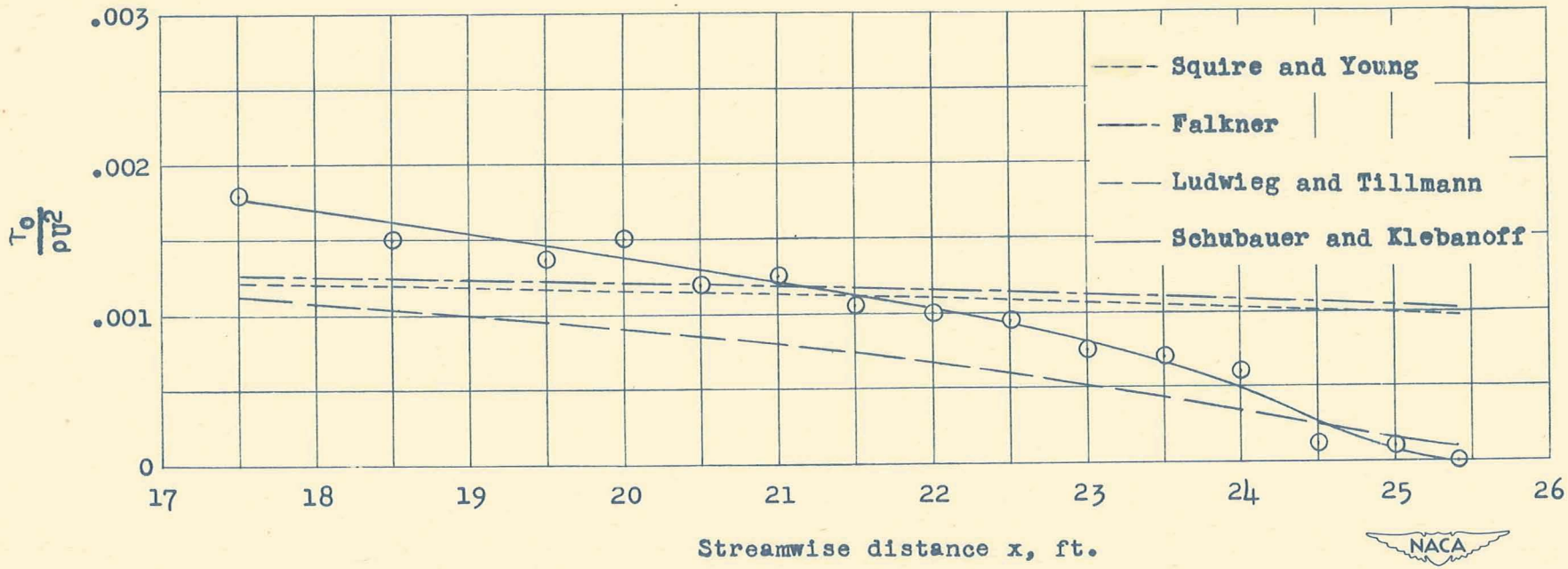

FIGURE 5

STREAMWISE VARIATION OF THE COEFFICIENT OF WALL SHEARING STRESS AS OBTAINED BY VARIOUS METHODS.

of Squire and Young has been used extensively in America in conjunction with the method of von Doenhoff and Tetervin[20] and the flat plate formula of Falkner has been used in England in conjunction with the method of Garner[21]. The formula of Ludwieg and Tillmann was the result of some experiments made to determine wall shearing stress in a pressure gradient with the aid of a calibrated heat-transfer instrument. Although the estimated values of wall shearing stress given by Schubauer and Klebanoff became zero at separation these values of wall shearing stress are considerably higher than those given by the flat plate formulas at lower values of x. These estimated values of wall shearing stress were obtained by fairing the $\rho\overline{u'v'}$ data to the region of the wall and then extrapolating the wall shearing stress by continuing the fairing to the wall with a slope equal to the pressure gradient. A positive pressure gradient tends to decrease the velocity in the boundary layer and hence decrease the normal gradient of velocity at the wall. Because the flow in the immediate neighborhood of the wall is laminar the decrease in velocity gradient at the wall is directly proportional to the decrease in wall shearing stress. For all values of x

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[20] Von Doenhoff and Tetervin, loc. cit.

[21] Garner, loc. cit.

larger than 17.5 the pressure gradient is positive so it would be expected that the wall shearing stress be less than that obtained by flat plate formulas.

The distribution of wall shearing stress obtained by using the formula of Ludwieg and Tillmann appears to be the most reasonable even though zero wall shearing stress is not indicated at separation. Inspection of Ludwieg and Tillmann's formula

$$\frac{\tau_o}{\bar{\rho} U^2} = \left(0.123\right)\left(\frac{1}{10^{0.678H}}\right)\left(\frac{1}{R_\theta^{0.268}}\right)$$

reveals that H must be infinite for the wall shearing stress to disappear and this is, of course, physically unreasonable. At separation, however, the indicated value of wall shearing stress is small enough to be considered negligible.

Substitution of data in the Karman momentum theorem. In order to evaluate the Karman momentum theorem as a momentum balance for turbulent boundary layers, values of $\partial\theta/\partial x$, and $(H + 2)\frac{\theta}{U}\frac{\partial U}{\partial x}$ were computed and plotted as a function of x in Figure 6. The wall shearing stress computed from the formula of Ludwieg and Tillmann is also plotted in Figure 6. If the Karman momentum theorem required only mean flow quantities, the sum of the ordinates would be zero. The sum of the ordinates is plotted as a function of x in Figure 7, and it can be seen that for a large range of x position this form of the Karman momentum theorem gives good results; near separation, however, considerable discrepancies exist. Good

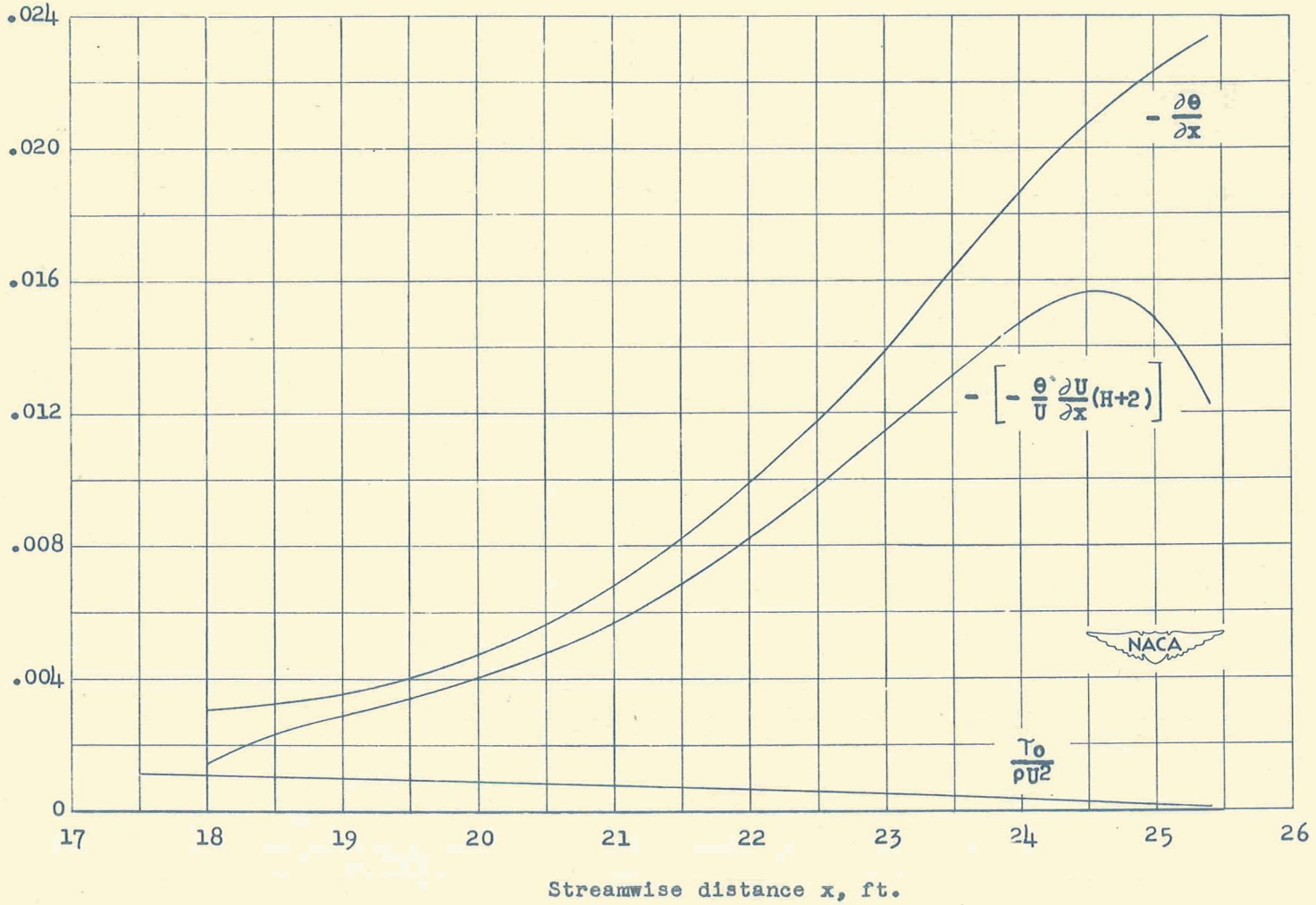

FIGURE 6

STREAMWISE VARIATION OF THE TERMS IN THE KARMAN MOMENTUM THEOREM AS COMPUTED FROM THE DATA OF SCHUBAUER AND KLEBANOFF.

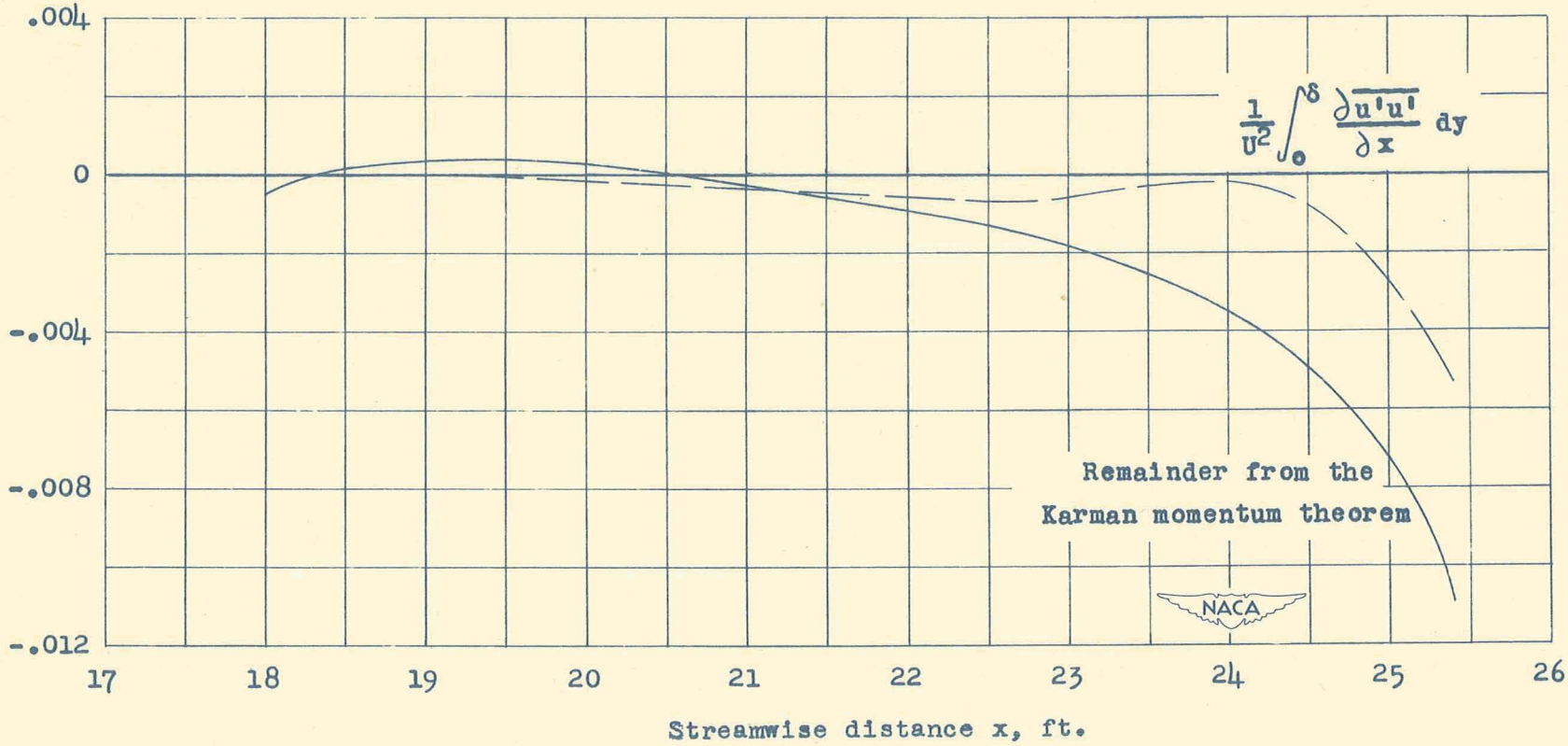

FIGURE 7

STREAMWISE VARIATION OF THE MOMENTUM REMAINDER FROM THE KARMAN MOMENTUM THEOREM AND THE MOMENTUM TERM DUE TO THE TURBULENT LONGITUDINAL FLUCTUATION.

results are obtained up to a value of H of approximately 1.5.

Because changes in the fairing of θ will alter the value of $\partial\theta/\partial x$, it is of interest to know how much different the values of θ would have to be to account for the noted discrepancies in this form of the Karman momentum theorem. The required $\partial\theta/\partial x$ was integrated with respect to x from 17.5 feet to 25.4 feet and the results are plotted as a dotted line in Figure 2 (page 19). It can be seen that the differences between the experimental curve and the required curve could not be attributed to differences in fairing.

II. FLUCTUATING-FLOW DATA

Presentation of data and discussion of accuracy.

Values of $\frac{\sqrt{\overline{u'u'}}}{U_m}$, reported by Schubauer and Klebanoff, have been plotted as a function of x and y in Figure 8; the data were plotted "three-dimensionally" to facilitate fairing. Successive planes of x = constant were displaced along a 45° axis proportional to the magnitude of x. By this method of plotting, curves of y = constant can be used to check the fairing and to aid in the visualization of the surface. This method of plotting and fairing can be used because the course of the fluctuating-flow data is regular up to separation.

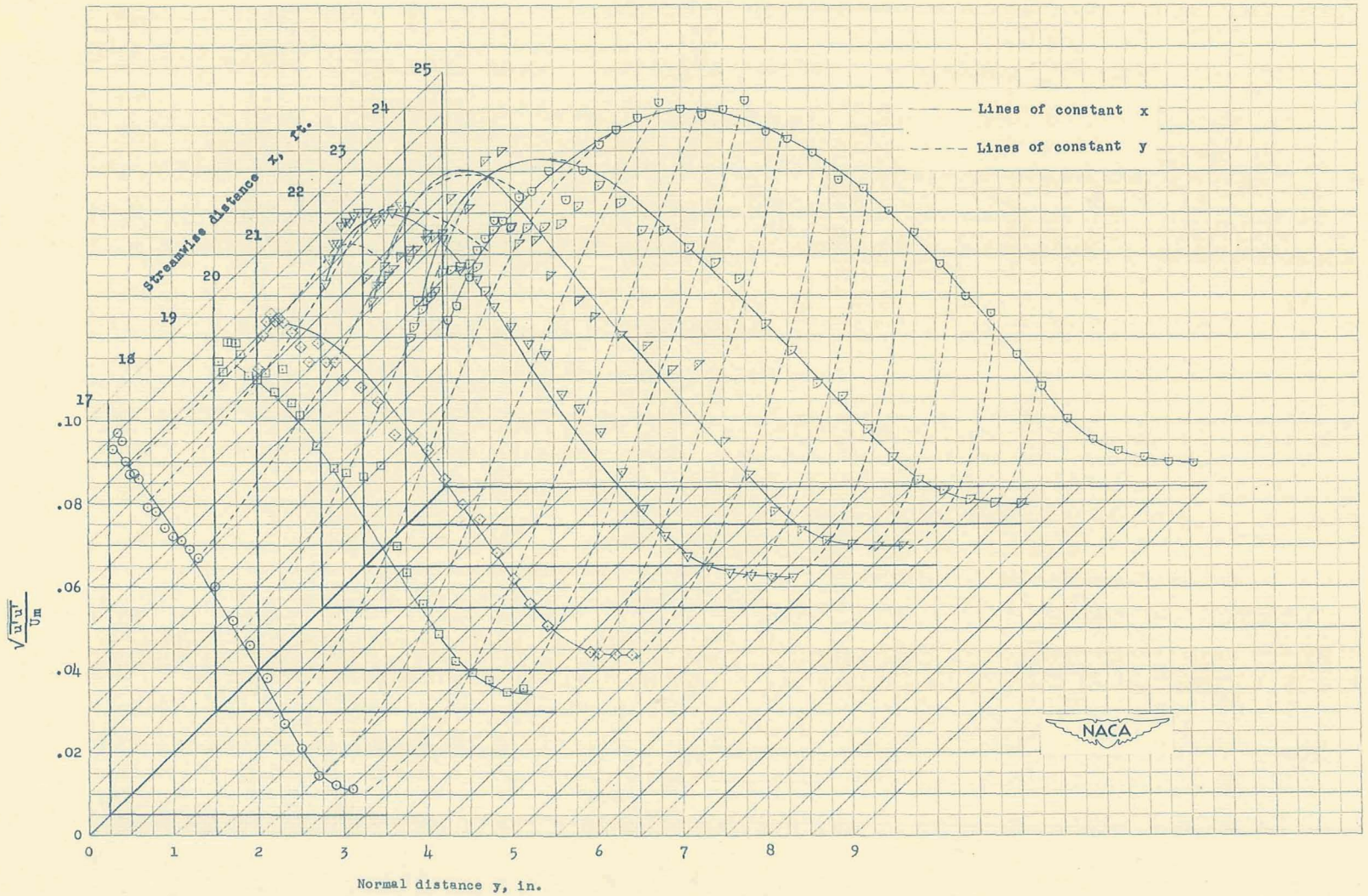

FIGURE 8

STREAMWISE AND NORMAL VARIATION OF THE TURBULENT LONGITUDINAL FLUCTUATION RATIO, $\frac{\sqrt{u'u'}}{U_m}$. DATA WERE OBTAINED FROM SCHUBAUER AND KLEBANOFF.

Scatter of the data indicates experimental random errors of about 10 percent for $\frac{\sqrt{\overline{u'u'}}}{U_m}$. It is believed that the "three-dimensional" method of plotting and fairing gives a considerable improvement in the net accuracy of the data.

Evaluation of the momentum terms due to turbulence. The faired data of Figure 8 (page 29) were differentiated with respect to x and integrated through the boundary layer; the resulting curve of $\frac{1}{U^2}\int_0^\delta \frac{\partial \overline{u'u'}}{\partial x} dy$ is plotted as a function of x in Figure 7 (page 27). The magnitude of this function is largely dependent on the fairing of the data in Figure 8 (page 29), but the data are sufficiently consistent to establish an order of magnitude.

A comparison of the values of $\frac{1}{U^2}\int_0^\delta \frac{\partial \overline{u'u'}}{\partial x} dy$ with the momentum remainder from the Karman momentum theorem, indicates that this term is not negligible. It may, indeed, be a large contributing factor in the Karman momentum theorem, but nothing more than a qualitative evaluation can be made with these data.

In order to evaluate the term

$$\frac{1}{U^2}\int_0^\delta\int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy\, dy$$

the data of Schubauer and Klebanoff were investigated. The differentiation and integration processes so greatly magnify the errors that the magnitude of the term may be completely changed by small changes in fairing. A calculation was made in which the integrand was believed to be large. The

calculation indicated that the term

$$\frac{1}{U^2}\int_0^{\delta}\int_0^{y}\frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x}+\frac{\partial \overline{v'v'}}{\partial y}\right)dy\,dy$$

was a small fraction of the term

$$\frac{1}{U^2}\int_0^{\delta}\frac{\partial \overline{u'u'}}{\partial x}dy$$

but the results were too inaccurate to warrant presentation.

Data are sufficient to give a magnitude to $\frac{\partial \overline{u'v'}}{\partial x}$ but are completely unable to permit calculation of $\frac{\partial^2 \overline{u'v'}}{\partial x^2}$. Because the factor δ appears in the numerator of the term

$$\frac{\delta}{U^2}\int_0^{\delta}\frac{\partial^2 \overline{u'v'}}{\partial x^2}dy$$

it is believed that the magnitude of the term is small.

III. SUGGESTIONS FOR FURTHER RESEARCH

It has been suggested that the expression $\frac{1}{U^2}\int_0^{\delta}\frac{\partial \overline{u'u'}}{\partial x}dy$ may be of sufficient magnitude to require its inclusion in the Karman momentum theorem for turbulent boundary layers. In order to establish the validity of this idea a complete study is required of the flow in the vicinity of separation. Particular attention must be given to accurate determination of the mean flow velocity profile and to a complete determination of $\frac{\overline{u'u'}}{U^2}$, $\frac{\overline{u'v'}}{U^2}$, and $\frac{\overline{v'v'}}{U^2}$ as functions of x and y. The study should include a range of pressure gradient and, for completeness, several different degrees of surface roughness.

CHAPTER V

CONCLUDING REMARKS

A reevaluation of the assumptions leading to the boundary-layer equations for turbulent flow indicates that certain derivatives of fluctuating quantities have been neglected. Experimental data show that the streamwise derivative of the $\overline{u'u'}$ fluctuation component may be of sufficient magnitude to require its inclusion when the Karman momentum theorem is used for turbulent boundary layers near separation. The data show that in this particular case the Karman momentum theorem gives satisfactory results without this term up to a value of H of approximately 1.5.

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APPENDIX

APPENDIX

The turbulent boundary layer equation is integrated with respect to y through the boundary layer or

$$\int_0^\delta \left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) dy + \int_0^\delta \left(\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y}\right) dy = \int_0^\delta U \frac{\partial U}{\partial x}$$

$$- \int_0^\delta \frac{\partial}{\partial x} \int_0^\delta \frac{\partial \overline{u'v'}}{\partial x} dy\, dy + \int_0^\delta \int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy\, dy + \int_0^\delta \nu \frac{\partial^2 \bar{u}}{\partial y^2} dy \qquad \text{(A1)}$$

Now

$$\int_0^\delta \bar{u}\frac{\partial \bar{u}}{\partial x} dy = \frac{1}{2}\frac{\partial}{\partial x}\int_0^\delta \bar{u}^2 dy - \frac{1}{2} U^2 \frac{\partial \delta}{\partial x}$$

$$\bar{v} = -\int_0^y \frac{\partial \bar{u}}{\partial x} dy$$

$$\int_0^\delta \nu \frac{\partial^2 \bar{u}}{\partial y^2} dy = \nu\left[\left(\frac{\partial \bar{u}}{\partial y}\right)_\delta - \left(\frac{\partial \bar{u}}{\partial y}\right)_0\right] = -\nu\left(\frac{\partial \bar{u}}{\partial y}\right)_0 = -\frac{\tau_0}{\rho}$$

$$\int_0^\delta \frac{\partial}{\partial x}\int_0^\delta \frac{\partial \overline{u'v'}}{\partial x} dy\, dy = \delta \int_0^\delta \frac{\partial^2 \overline{u'v'}}{\partial x^2} dy$$

If these quantities are substituted in equation (A1), there is obtained the relation

$$\frac{1}{2}\frac{\partial}{\partial x}\int_0^\delta \bar{u}^2 dy - \int_0^\delta \left(\int_0^y \frac{\partial \bar{u}}{\partial x} dy\right)\left(\frac{\partial \bar{u}}{\partial y}\right) dy = \frac{1}{2}\frac{\partial}{\partial x}\int_0^\delta U^2 dy - \frac{\tau_0}{\rho}$$

$$- \int_0^\delta \frac{\partial \overline{u'u'}}{\partial x} dy - \delta \int_0^\delta \frac{\partial^2 \overline{u'v'}}{\partial x^2} dy + \int_0^\delta \int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy\, dy \qquad \text{(A2)}$$

where

$$\int_0^\delta \frac{\partial \overline{u'v'}}{\partial y} dy = (\overline{u'v'})_\delta - (\overline{u'v'})_0 = 0$$

Now

$$\int_0^\delta \left(\int_0^y \frac{\partial \bar{u}}{\partial x} dy \right) \left(\frac{\partial \bar{u}}{\partial y} \right) dy = U \frac{\partial}{\partial x} \int_0^\delta \bar{u}\, dy - \frac{1}{2} \cdot \frac{\partial}{\partial x} \int_0^\delta \bar{u}^2 dy \qquad \text{(A3)}$$

Substitute equation (A3) in equation (A2) and divide by U^2, then

$$\frac{1}{U^2} \frac{\partial}{\partial x} \int_0^\delta \bar{u}^2 dy - \frac{1}{U} \frac{\partial}{\partial x} \int_0^\delta \bar{u}\, dy = \frac{1}{2} \frac{1}{U^2} \frac{\partial}{\partial x} \int_0^\delta U^2 dy - \frac{\tau_0}{\rho U^2}$$

$$- \frac{1}{U^2} \int_0^\delta \frac{\partial \overline{u'u'}}{\partial x} dy - \frac{\delta}{U^2} \int_0^\delta \frac{\partial^2 \overline{u'v'}}{\partial x^2} dy + \frac{1}{U^2} \int_0^\delta \int_0^y \frac{\partial}{\partial x} \left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} \right) dy\, dy \qquad \text{(A4)}$$

or by adding and subtracting $\frac{1}{2} \frac{1}{U^2} \frac{\partial}{\partial x} \int_0^\delta U^2 dy$

$$\frac{1}{U^2} \frac{\partial}{\partial x} \int_0^\delta (\bar{u}^2 - U^2) dy - \frac{1}{U} \frac{\partial}{\partial x} \int_0^\delta (\bar{u} - U) dy + \frac{\tau_0}{\rho U^2} = - \frac{1}{U^2} \int_0^\delta \frac{\partial \overline{u'u'}}{\partial x} dy$$

$$- \frac{\delta}{U^2} \int_0^\delta \frac{\partial^2 \overline{u'v'}}{\partial x^2} dy + \frac{1}{U^2} \int_0^\delta \int_0^y \frac{\partial}{\partial x} \left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} \right) dy\, dy \qquad \text{(A5)}$$

Let the momentum loss M of the stream be defined by

$$M = \rho U^2 \theta$$

where θ is the momentum thickness, then

$$\bar{\rho} U^2 \theta = \bar{\rho} \int_0^{\delta} (\bar{u} U - \bar{u}^2)\, dy$$

or

$$\theta = \int_0^{\delta} \frac{\bar{u}}{U} \left(1 - \frac{\bar{u}}{U}\right) dy$$

Now

$$\frac{\partial \theta}{\partial x} = -\frac{1}{U^2} \frac{\partial}{\partial x} \int_0^{\delta} (\bar{u}^2 - U^2)\, dy + \frac{1}{U} \frac{\partial}{\partial x} \int_0^{\delta} (\bar{u} - U)\, dy - \frac{1}{U} \frac{dU}{dx} (2\theta + \delta_*) \qquad \text{(A6)}$$

where δ_* the displacement thickness, is defined by

$$\delta_* = \int_0^{\delta} \left(1 - \frac{\bar{u}}{U}\right) dy$$

By using equation (A6), equation (A5) may be written finally as

$$\frac{\tau_0}{\bar{\rho} U^2} - \frac{\partial \theta}{\partial x} - (H+2) \frac{\theta}{U} \frac{\partial U}{\partial x} = -\frac{1}{U^2} \int_0^{\delta} \frac{\partial \overline{u'u'}}{\partial x}\, dy$$

$$+ \frac{1}{U^2} \int_0^{\delta} \int_0^{y} \frac{\partial}{\partial x} \left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} \right) dy\, dy - \frac{\delta}{U^2} \int_0^{\delta} \frac{\partial^2 \overline{u'v'}}{\partial x^2}\, dy$$

where H is defined as δ_*/θ.

This equation is the momentum theorem for turbulent boundary layers (compare equation 11).